FINAL: RINGS AND MODULES

Date: **4th May 2026**
The Total points is **110**. The maximum you can score is 100.

A **ring** would mean a **commutative ring with identity**.

(1) (3+6+6+6=21 points) Define noetherian rings. Prove or disprove the following.
   (a) If R_1 and R_2 are noetherian then so is $R_1 \times R_2$.
   (b) If R_1 is noetherian and R_2 is a subring of R_1 then R_2 is noetherian.
   (c) If $q : R_1 \to R_2$ is a surjective ring homomorphism and R_1 is noetherian then R_2 is noetherian.

(2) (4+5+8+5=22 points) For an R module M define the set of torsion elements of M and $Ann_R(M)$. Let $R = (\mathbb{Z}/12\mathbb{Z})[x]$ be the polynomial ring over $\mathbb{Z}/12\mathbb{Z}$. Compute the nilradical of R. Prove or disprove the following statement. Every nonzero R-module contains a nonzero torsion element. Also compute $Ann_R(M)$ where $R = \mathbb{Z}$ and $M = \mathbb{Z}/12\mathbb{Z} \oplus \mathbb{Z}/18\mathbb{Z}$.

(3) (7+7+7=21 points) Give a counter example to disprove the following statements.
   (a) For a ring R, every torsion module has a nonzero annihilator.
   (b) Every torsion free $\mathbb{Z}$-module is free.
   (c) Every finitely generated module over any ring is a noetherian module.

(4) (4+7+10=21 points) Let R be a ring and S a multiplicative subset. Define the localization $S^{-1}R$ of the ring R. Consider the following rings.
   (a) $A = (\mathbb{Z}/45\mathbb{Z})[x, y, z]/(3x - 1, z^3x - x^2y + x^3y)$
   (b) $B = (\mathbb{Z}/5\mathbb{Z})[x, y, z]/(3x - 1, z^3x - x^2y + x^3y)$
   (c) $C = \mathbb{Z}[x, y, z]/(3x - 1, z^3x - x^2y + x^3y)$

   Show that A and B are isomorphic. Show that these rings are UFD and determine if any of them are a PID.

(5) (5+15+5=25 points) State the structure theorem for finitely generated modules over a PID. Let $V_1 = \mathbb{C}[y]/((y^2 + 1)(y^4 - 1))$, $V_2 = \mathbb{C}[z]/(z^3 - z^2)$ and $V = V_1 \oplus V_2$ be $\mathbb{C}$-vector spaces. Let $\phi_1 : V_1 \to V_1$, $\phi_2 : V_2 \to V_2$ and $\phi : V \to V$ be given by $\phi_1(v_1) = \bar{y}v_1$, $\phi_2(v_2) = \bar{z}v_2$ and $\phi(v_1, v_2) = (\bar{y}v_1, \bar{z}v_2)$ $\forall v_1 \in V_1, \forall v_2 \in V_2$ respectively. Find the rational canonical forms and the Jordan forms of ϕ_1, ϕ_2 and ϕ. Compute the minimal polynomial of ϕ?